Attachment 2

Balancing the Type I and Type II Error Probabilities of the Modified Z Test Statistic

This paper describes the methodology for balancing the error probabilities when the Modified Z statistic is used for performance measure parity testing. There are four key elements of the statistical testing process:

1. the null hypothesis, H_0, that parity exists between ILEC and CLEC services,
2. the alternative hypothesis, H_a, that the ILEC is giving better service to its own customers,
3. the Modified Z test statistic, Z, and
4. a critical value, c.

The decision rule[1] is

- If $Z < c$, then accept H_a.
- If $Z \geq c$, then accept H_0.

There are two types of error possible when using such a decision rule:

Type I Error: Deciding favoritism exists (accept H_a) when there is, in fact, no favoritism (H_0 is true).
Type II Error: Deciding parity exists (accept H_0) when there is, in fact, favoritism (H_a is true).

The probabilities of the two types of error are:

Type I Error: $\alpha = P(Z < c \mid H_0)$.
Type II Error: $\beta = P(Z \geq c \mid H_a)$.

In what follows, we show how to find a balancing critical value, c_B, so that $\alpha = \beta$.

General Methodology

The general form of the test statistic that is being used is

$$z_0 = \frac{\hat{T} - E(\hat{T} \mid H_0)}{SE(\hat{T} \mid H_0)}, \qquad (1)$$

[1] This decision rule assumes that the smaller a performance measure is, the better the service. If the opposite is true, then the decision rule should be reversed by using $-Z$ in place of Z.

where

> $\hat{T}$ is an estimator that is (approximately) normally distributed,
>
> $E(\hat{T} \mid H_0)$ is the expected value (mean) of $\hat{T}$ under the null hypothesis, and
>
> $SE(\hat{T} \mid H_0)$ is the standard error of $\hat{T}$ under the null hypothesis.

Thus, under the null hypothesis, z_0 follows a standard normal distribution. However, this is not true under the alternative hypothesis. In this case,

$$z_a = \frac{\hat{T} - E(\hat{T} \mid H_a)}{SE(\hat{T} \mid H_a)}$$

has (approximately) a standard normal distribution. Here

> $E(\hat{T} \mid H_a)$ is the expected value (mean) of $\hat{T}$ under the alternative hypothesis, and
>
> $SE(\hat{T} \mid H_a)$ is the standard error of $\hat{T}$ under the alternative hypothesis.

Notice that

$$\begin{aligned}\beta &= \mathrm{P}(z_0 > c \mid H_a) \\ &= \mathrm{P}\left(z_a > \frac{cSE(\hat{T} \mid H_0) + E(\hat{T} \mid H_0) - E(\hat{T} \mid H_a)}{SE(\hat{T} \mid H_a)} \right),\end{aligned} \qquad (2)$$

and recall that for a standard normal random variable z and a constant b, $\mathrm{P}(z < b) = \mathrm{P}(z > -b)$. Thus,

$$\alpha = \mathrm{P}(z_0 < c) = \mathrm{P}(z_0 > -c). \qquad (3)$$

Since we want $\alpha = \beta$, the right hand sides of (2) and (3) represent the same area under the standard normal density. Therefore, it must be the case that

$$-c = \frac{cSE(\hat{T} \mid H_0) + E(\hat{T} \mid H_0) - E(\hat{T} \mid H_a)}{SE(\hat{T} \mid H_a)}.$$

Solving this for c give the general formula for a balancing critical value:

$$c_B = \frac{E(\hat{T} \mid H_a) - E(\hat{T} \mid H_0)}{SE(\hat{T} \mid H_a) + SE(\hat{T} \mid H_0)}. \qquad (4)$$

The Balancing Critical Value of the Modified Z for a Mean Measure

The modified Z statistic, *Z*, for a mean measure is given by

$$Z = \frac{\hat{T}}{s_1\sqrt{1/n_1 + 1/n_2}},$$

where $\hat{T} = \overline{X}_1 - \overline{X}_2$, and subscripts 1 and 2 refer to ILEC and CLEC quantities, respectively.

One possible set of hypotheses that take into account the assumption that transaction are identically distributed within LECs, is:

H_0: $\mu_1 = \mu_2$, $\sigma_1^2 = \sigma_2^2$,

H_a: $\mu_2 = \mu_1 + \delta\cdot\sigma_1$, $\sigma_2^2 = \lambda\cdot\sigma_1^2$, where $\delta > 0$ and $\lambda \geq 1$.

Assuming that n_1 is large enough so that s_1 adequately approximates σ_1, we have

$$E(\hat{T} \mid H_0) = 0,$$

$$SE(\hat{T} \mid H_0) = \sigma_1\sqrt{1/n_1 + 1/n_2},$$

$$E(\hat{T} \mid H_a) = -\delta\sigma_1,$$

$$SE(\hat{T} \mid H_a) = \sigma_1\sqrt{1/n_1 + \lambda/n_2}.$$

Substituting these values in equation (4) gives

$$c_B = \frac{-\delta}{\sqrt{1/n_1 + 1/n_2} + \sqrt{1/n_1 + \lambda/n_2}}$$

$$= \frac{-\delta\sqrt{n_1 n_2}}{\sqrt{n_1 + n_2} + \sqrt{\lambda n_1 + n_2}}.$$

The preceding equations have indexed the alternative hypothesis by two parameters, λ and δ. While statistical science can be used to evaluate the impact of different choices of these parameters, there is not much that an appeal to statistical principles can offer in directing specific choices. Specific choices are best left to telephony experts. Still, it is possible to comment on some aspects of these choices:

Parameter Choice for λ. The parameter λ indexes an alternative to the null hypothesis that arises because there might be greater unpredictability or variability in the delivery of service to a CLEC customer over that which would be achieved for an otherwise comparable ILEC customer. Typically, there is little basis for choosing a value of λ other than 1, in which case the formula for c_B simplifies to

$$c_B = \frac{-\delta\sqrt{n_1 n_2}}{2\sqrt{n_1 + n_2}}.$$

Parameter Choice for δ. The parameter δ is much more important in the choice of the balancing point than was true for λ because it directly indexes the difference in average service.

The Balancing Critical Value of the Modified Z for a Proportion Measure

Specification of a balancing critical value for a proportion measure is more complex than for mean measures because c_B depends directly on both the assumed ILEC and CLEC proportions under H_a not just through a single parameter like δ.

The modified Z statistic for a proportion measure is given by

$$Z = \frac{\hat{T}}{\sqrt{\hat{p}_{ILEC}(1-\hat{p}_{ILEC})}\sqrt{1/n_1 + 1/n_2}},$$

where $\hat{T} = \hat{p}_{ILEC} - \hat{p}_{CLEC}$, and where n_1 and n_2 are the ILEC and CLEC sample sizes, respectively.

The null and alternative hypotheses are specified fully in terms of the true proportions p_{ILEC} and p_{CLEC} as follows:

H_0: $p_{ILEC} = p_{CLEC} = p_1$,

H_a: $p_{ILEC} = p_1$, $p_{CLEC} = p_2 > p_1$.

Assuming that n_1 is large enough so that $\hat{p}_{ILEC}(1-\hat{p}_{ILEC})$ adequately approximates $p_{ILEC}(1-p_{ILEC})$, then Z satisfies (1) and we have

$$E(\hat{T} \mid H_0) = 0,$$

$$SE(\hat{T} \mid H_0) = \sqrt{p_1(1-p_1)}\sqrt{1/n_1 + 1/n_2},$$

$$E(\hat{T} \mid H_a) = p_1 - p_2,$$

$$SE(\hat{T} \mid H_a) = \sqrt{p_1(1-p_1)/n_1 + p_2(1-p_2)/n_2}.$$

Substituting these values in equation (4) gives

$$c_B = \frac{-(p_2 - p_1)}{\sqrt{p_1(1-p_1)/n_1 + p_2(1-p_2)/n_2} + \sqrt{p_1(1-p_1)}\sqrt{1/n_1 + 1/n_2}}.$$

A convenient way to specify the alternative hypothesis is through the "odds ratio" for p_2 and p_1; specifically

$$\varphi = \left(\frac{p_2}{p_1}\right)\left(\frac{1-p_1}{1-p_2}\right),$$

so that

$$p_2 = \left(\frac{\varphi\, p_1}{1 + (\varphi - 1)p_1}\right).$$

The Balancing Critical Value of the Modified Z for a Rate Measure

A rate is a ratio of two counts *num/denom*—e.g., $r_{ILEC} = num_{ILEC}/denom_{ILEC}$. Where the *denom* count is assumed known but the *num* count is subject to sampling variability. Similarly to proportions, the balancing critical value c_B depends directly on the assumed ILEC and CLEC rates under H_a as well as the ILEC and CLEC denominators.

The modified Z statistic for a rate measure is given by

$$Z = \frac{\hat{T}}{\sqrt{\hat{r}_{ILEC}(1/denom_{CLEC} + 1/denom_{ILEC})}}.$$

Where $\hat{T} = \hat{r}_{ILEC} - \hat{r}_{CLEC}$.

The null and alternative hypotheses are specified fully in terms of the true proportions r_{ILEC} and r_{CLEC} as follows:

$$H_0: \; r_{ILEC} = r_{CLEC} = r_1 \,,$$

$$H_a: r_{ILEC} = r_1, \; r_{CLEC} = r_2 > r_1.$$

Assuming that $denom_{ILEC}$ is large enough so that $\hat{r}_{ILEC}$ adequately approximates r_{ILEC}, then Z satisfies (1) and we have

$$E(\hat{T} \mid H_0) = 0,$$

$$SE(\hat{T} \mid H_0) = \sqrt{r_{ILEC}(1/denom_{CLEC} + 1/denom_{ILEC})},$$

$$E(\hat{T} \mid H_a) = r_1 - r_2,$$

$$SE(\hat{T} \mid H_a) = \sqrt{r_{CLEC}/denom_{CLEC} + r_{ILEC}/denom_{ILEC}}\,.$$

Substituting these values in equation (4) gives

$$c_B = \frac{-(r_2 - r_1)}{\sqrt{r_{CLEC}/denom_{CLEC} + r_{ILEC}/denom_{ILEC}} + \sqrt{r_{ILEC}(1/denom_{CLEC} + 1/denom_{ILEC})}}\,.$$

A convenient way to specify the alternative hypothesis is by

$$r_2 = \varepsilon\, r_1 \,.$$